Oscillations

Oscillating Systems

Each day we encounter many kinds of oscillatory motion, *such as swinging pendulum of a clock, a person bouncing on a trampoline, a vibrating guitar string, and a mass on a spring.*

They have common properties:

1. The particle oscillates back and forth about a equilibrium position. *The time necessary for one complete cycle (a complete repetition of the motion) is called* the period T.

2. No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position. Such a force is called a "restoring force".

3. The number of cycles per unit time is called the "frequency" f.

Unit: period (s)

$$f = \frac{1}{T}$$

frequency(Hz, SI unit), 1 Hz = 1 cycle/s

4. The magnitude of the maximum displacement from equilibrium is called the amplitude of the motion.

The simple harmonic oscillator and its motion

1. Simple harmonic motion

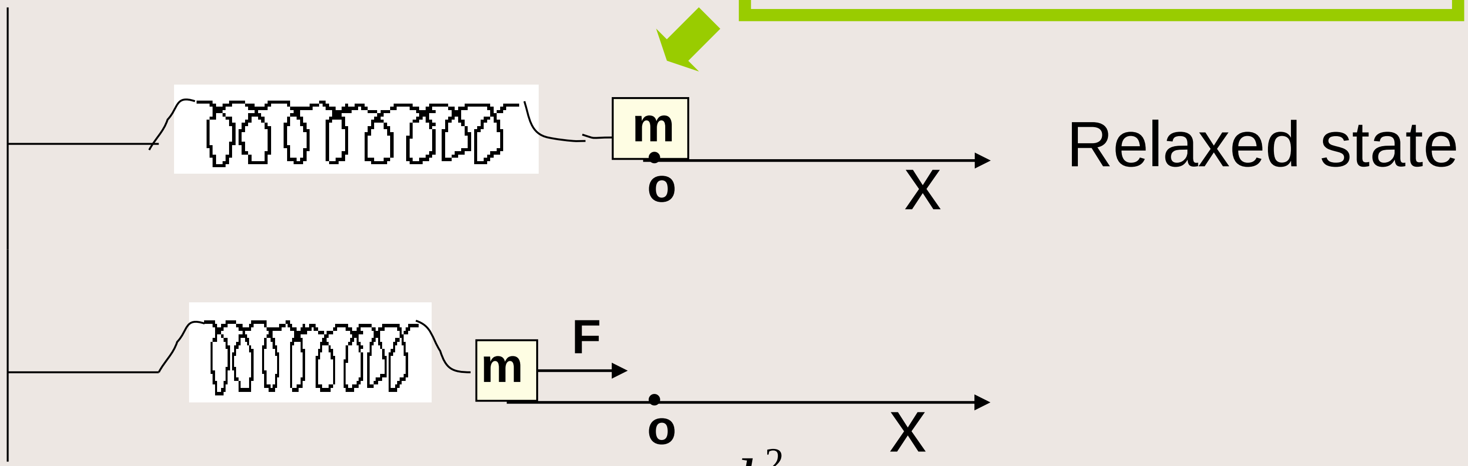
An oscillating system which can be described in terms of sine and cosine functions is called a "simple harmonic oscillator" and its motion is called "simple harmonic motion".

2. Equation of motion of the simple harmonic oscillator

Fig 17-5 shows a simple harmonic oscillator, consisting of a spring of force constant K acting on a body of mass m that slides on a frictionless horizontal surface. The body moves in x direction.

Fig 17-5

origin is chosen at here



$$\sum F_x = -kx \qquad a_x = \frac{d^2x}{dt^2}$$

$$\Rightarrow \quad -kx = m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad (17\text{-}4)$$

Eq(17-4) is called the "equation of motion of the simple harmonic oscillator". It is the basis of many complex oscillator problems.

3. Find the solution of Eq. (17-4)

Rewrite Eq(17-4) as

$$\frac{d^2x}{dt^2} = -(\frac{k}{m})x \qquad (17\text{-}5)$$

We write a tentative solution to Eq(17-5) as

$$x = x_m \cos(\omega t + \phi) \qquad (17\text{-}6)$$

We differentiate Eq(17-6) twice with respect to the time.

$$\frac{d^2x}{dt^2} = -\omega^2 x_m \cos(\omega t + \phi)$$

Putting this into Eq(17-5) we obtain

$$-\omega^2 x_m \cos(\omega t + \phi) = -\frac{k}{m} x_m \cos(\omega t + \phi)$$

Therefore, if we choose the constant ω such that

$$\omega^2 = \frac{k}{m} \qquad (17\text{-}7)$$

Eq(17-6) is in fact a solution of the equation of motion of a simple harmonic oscillator.

a) ω :

If we increase the time by $\frac{2\pi}{\omega}$ in Eq(17-6), then

$$x = x_m[\cos\omega(t + \frac{2\pi}{\omega}) + \phi] = x_m \cos(\omega t + \phi)$$

Therefore $\frac{2\pi}{\omega}$ is the period of the motion T.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \quad (17\text{-}8)$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad (17\text{-}9)$$

The quantity ω is called the angular frequency.

$$\omega = 2\pi f$$

b) x_m :

x_m is the maximum value of displacement. We call it the amplitude of the motion.

c) $\omega t + \phi$ and ϕ:

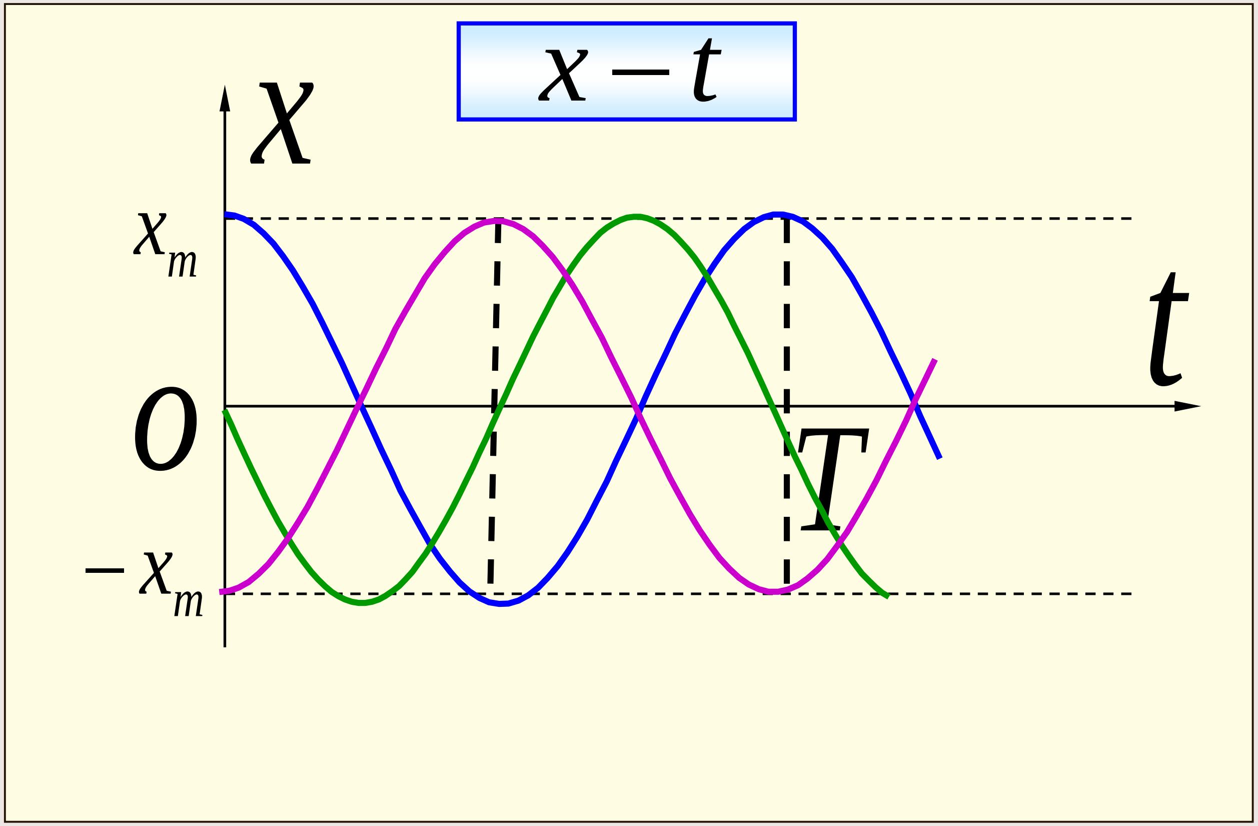
The quantity $\omega t + \phi$ is called phase of the motion.

ϕ is called "phase constant".

x_m and ϕ are determined by the initial position and velocity of the particle. ω is determined by the system.

How to understand ϕ ?

$$x = x_m \cos(\omega t + \varphi)$$



$\varphi = 0$

$\varphi = \frac{\pi}{2}$

$\varphi = \pi$

How to compare the phases of two SHOs with same ω ?

$$\begin{cases} x_1 = x_{m1}\cos(\omega t+\varphi_1) \\ x_2 = x_{m2}\cos(\omega t+\varphi_2) \end{cases} \Rightarrow \Delta\varphi = (\omega t+\varphi_2)-(\omega t+\varphi_1)$$

$$\Delta\varphi = \varphi_2 - \varphi_1$$

$\Delta\varphi = 0$	$\Delta\varphi = \pm\pi$	$\Delta\varphi$

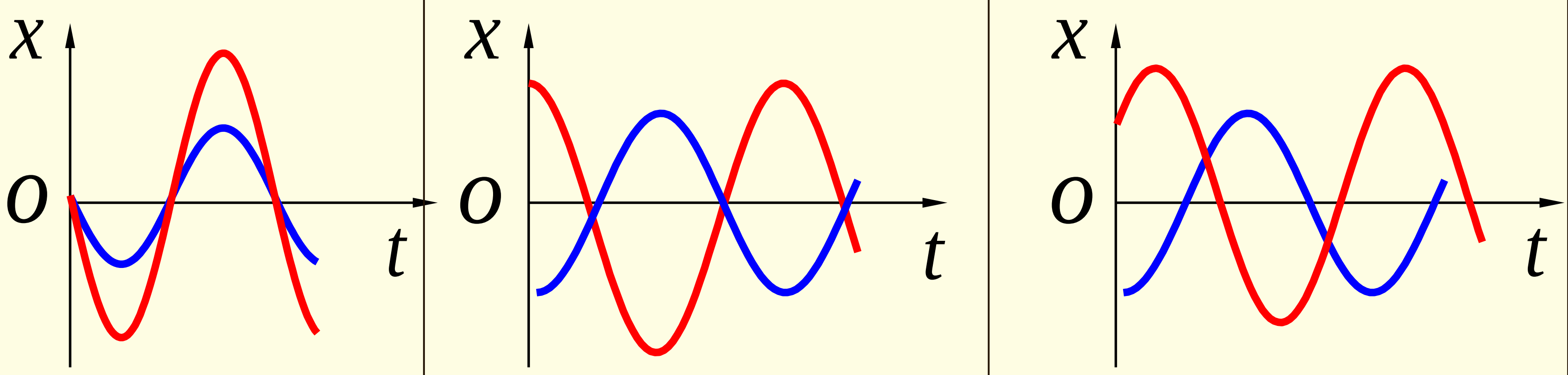
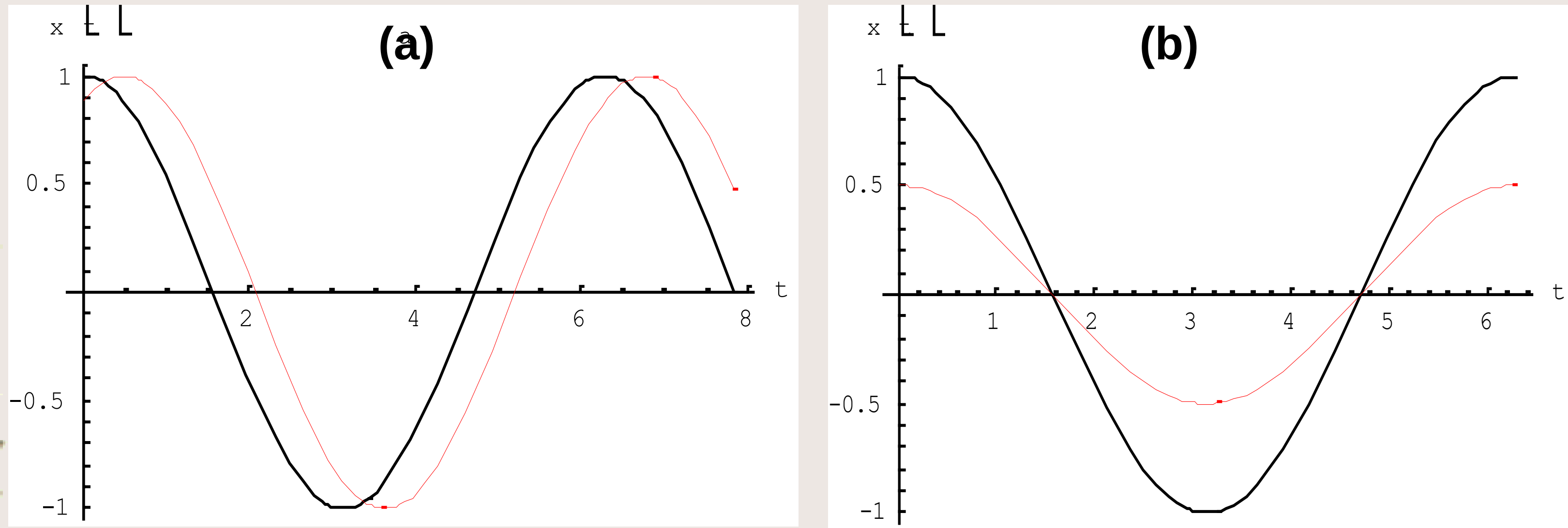
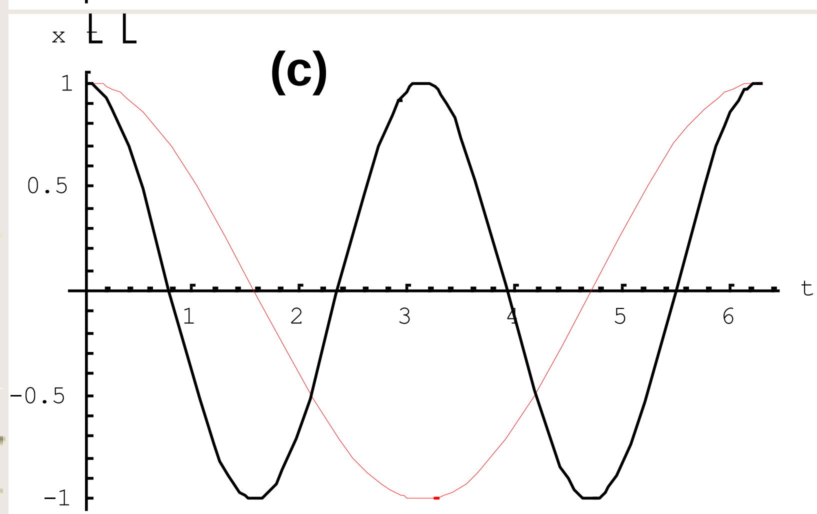


Fig 17-6 shows several simple harmonic motions.

Fig 17-6





(a) same: x_m , ω
different: ϕ

(b) same: ω , ϕ
different: x_m

(c) same: x_m , ϕ
different: ω

d). Displacement, velocity, and acceleration

Displacement $x = x_m \cos(\omega t + \phi)$

Velocity $v_x = \frac{dx}{dt} = -\omega x_m \sin(\omega t + \phi) = \omega x_m \cos(\omega t + \frac{\pi}{2} + \phi)$

Acceleration $a_x = \frac{d^2 x}{dt^2} = -\omega^2 x_m \cos(\omega t + \phi)$

$$= \omega^2 x_m \cos(\omega t + \pi + \phi) \qquad (17\text{-}11)$$

When the displacement is a maximum in either direction, the speed is zero, because the velocity must now change its direction.

$$x = x_m \cos(\omega t + \varphi)$$

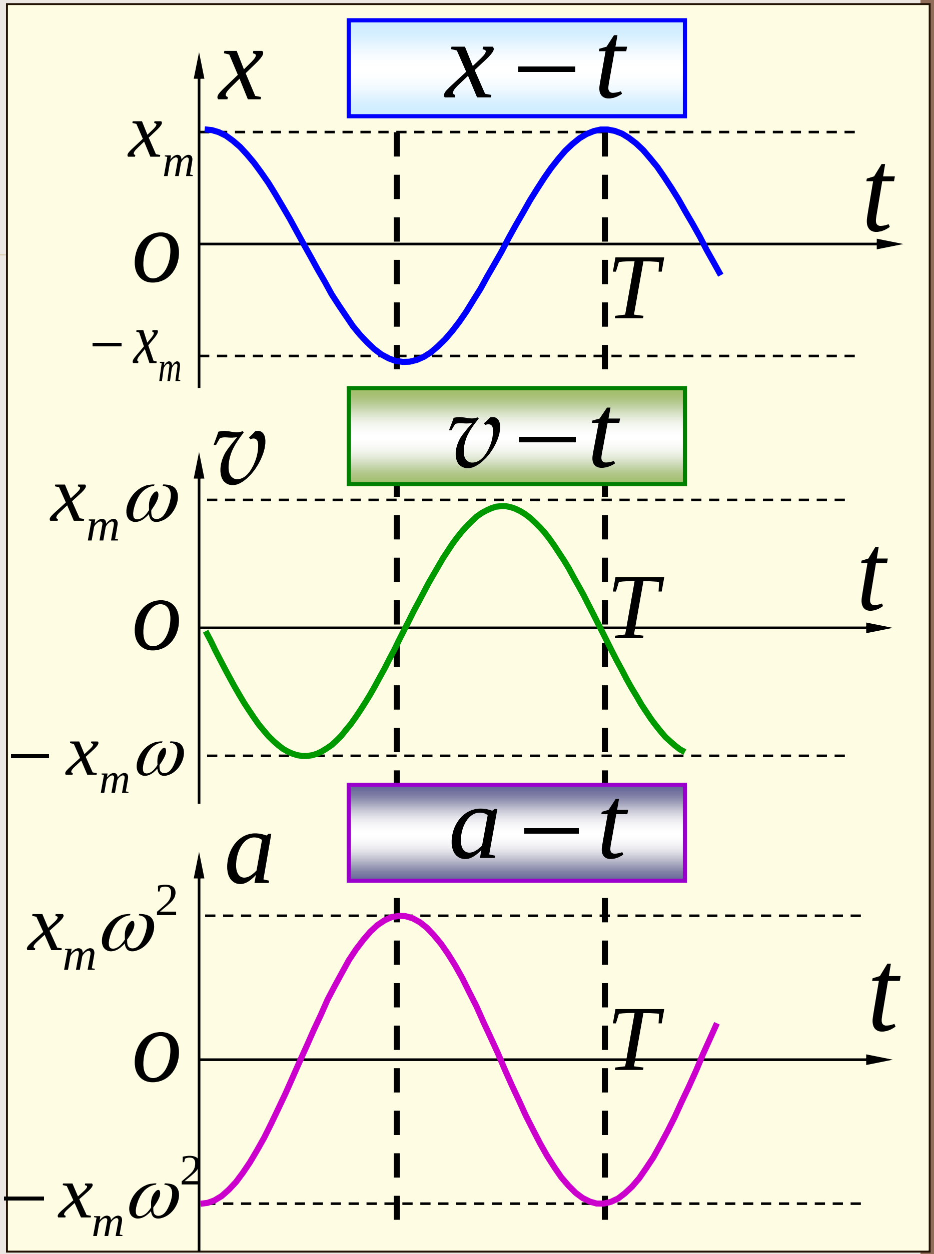
$$T = \frac{2\pi}{\omega} \qquad \varphi = 0$$

$$v = -x_m \omega \sin(\omega t + \varphi)$$

$$= x_m \omega \cos(\omega t + \varphi + \frac{\pi}{2})$$

$$a = -x_m \omega^2 \cos(\omega t + \varphi)$$

$$= x_m \omega^2 \cos(\omega t + \varphi + \pi)$$



Energy in simple harmonic motion

1.The potential energy

$$U=\frac{1}{2}kx^2=\frac{1}{2}kx_m{}^2\cos^2(\omega t+\phi)$$

(17-12)

2.The kinetic energy

$$K=\frac{1}{2}mv^2=\frac{1}{2}m\omega^2x_m{}^2\sin^2(\omega t+\phi)$$

$$=\frac{1}{2}kx_m{}^2\sin^2(\omega t+\phi)$$

(17-13)

$$v=-x_m\omega\sin(\omega t+\varphi)$$

$\phi=0$

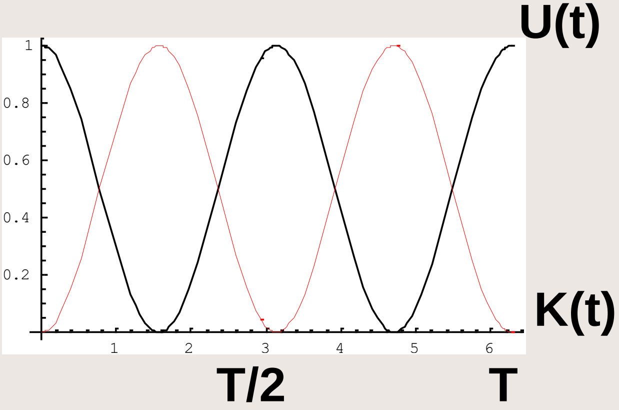


Fig 17-8(a)

- Fig17-8(a), both potential and kinetic energies oscillate with time *t* and vary between zero and maximum value of $\frac{1}{2}kx_m^{\ 2}$.

- Both U and K vary with twice the frequency of the the displacement and velocity.

3. The total mechanical energy E is

$$E = K + U = \frac{1}{2}kx_m^{\ 2} \qquad (17\text{-}14)$$

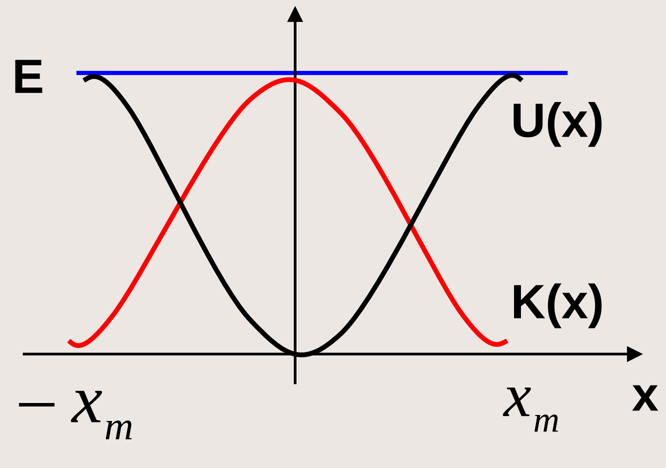


Fig 17-8 (b)

$$U(x) = \frac{1}{2}kx^2$$

$$K(x) = E - U(x)$$

At the maximum displacement $K = 0$, $U = \frac{1}{2}kx_m^2$

At the equilibrium position $U = 0$, $K = \frac{1}{2}kx_m^2$

Eq(17-14) can be written quite generally as

$$K + U = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \qquad (17\text{-}15)$$

then $$v_x^2 = \frac{k}{m}(x_m^2 - x^2)$$

or $$v_x = \pm\sqrt{\frac{k}{m}(x_m^2 - x^2)} \qquad (17\text{-}16)$$

Sample problem 17-2

Fig 17-5

In Fig 17-5, $m=2.43kg$, $k=221Nm^{-1}$, the block is stretched in the positive x direction a distance of *11.6 cm* from equilibrium and released. Take time $t=0$ when the block is released, the horizontal surface is frictionless.

(a) What is the total energy?

(b) What is the maximum speed of the block?

(c) What is the maximum acceleration?

(d) What is the position, velocity, and acceleration at $t=0.215s$?

Solution:

(a) $E = \frac{1}{2}kx_m^{\ 2} = \frac{1}{2}(221N/m)(0.116m)^2 = 1.49J$

(b)

$$v_{\max} = \sqrt{\frac{2K_{\max}}{m}} = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2(1.49J)}{2.43kg}} = 1.11m/s$$

(c) The maximum acceleration occurs just at the instant of release, when the force is greatest

$$a_{\max} = \frac{F_{\max}}{m} = \frac{kx_m}{m} = \frac{(221N/m)(0.116m)}{2.43kg} = 10.6m/s^2$$

(d) $\omega = \sqrt{\frac{k}{m}} = 0.9536 rad/s$

$$x(t) = x_m \cos(\omega t + \phi)$$

Since $x = x_m = 0.116m$ at $t=0$, then $\phi = 0$

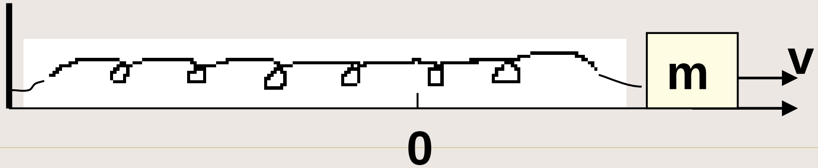
$$x(t) = x_m \cos \omega t = 0.116 \cos(9.536t)$$

So at $t=0.215s$

$$\begin{cases} x = 0.116 \cos(9.536)(0.215s) = -0.0535m \\ v_x = -\omega x_m \sin \omega t = -0.981m/s \\ a_x = -\omega^2 x = -(9.536rad/s)^2(-0.0535m) = 4.87m/s^2 \end{cases}$$

Sample problem 17-3

Fig 17-5



In Fig17-5, *m=2.43kg*, *k=221N/m*, when the block *m* is pushed from equilibrium to *x=0.0624m*, and its velocity $v_x = 0.847 m/s$, the external force is removed and the block begins to oscillate on the horizontal frictionless surface. Write an equation for $x(t)$ during the oscillation.

Solution: $x(t) = x_m cos(\omega t + \varphi), \quad x_m, \omega, \varphi$???

$$\omega = \sqrt{\frac{k}{m}} = 9.54 rad/s$$

x_m : At t=0

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$= \frac{1}{2}(2.43kg)(0.847m/s)^2 + \frac{1}{2}(221N/m)(0.0624m)^2$$

$$= 0.872J + 0.430J = 1.302J$$

Setting this equal to $\frac{1}{2}kx_m^2$, we have $x_m = \sqrt{\frac{2E}{k}} = 0.1085m$

To find the phase constant ϕ , we still need to use the information give for *t=0*:

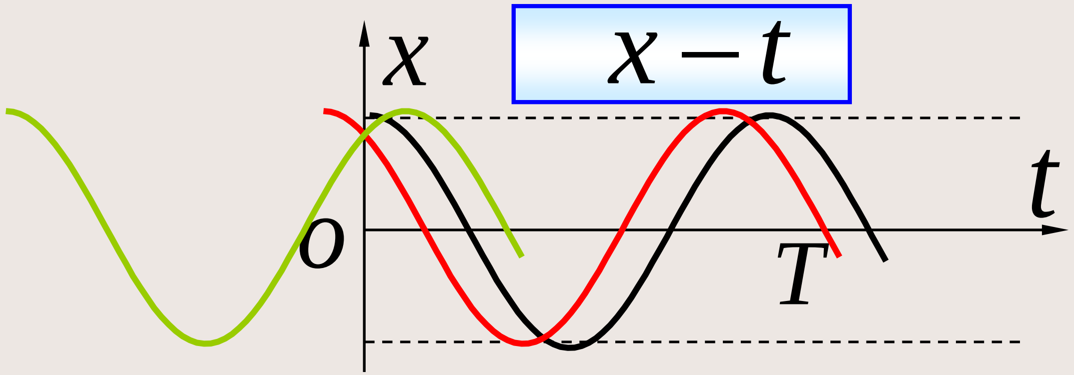
$$x(t) = x_m cos(\omega t + \varphi)$$
$$v_x = 0.847 m/s$$

$$x(0) = x_m \cos\phi = 0.0624 m$$

$$\cos\phi = \frac{x(0)}{x_m} = 0.5751 \quad \Rightarrow \quad \phi = \pm 54.9^{\circ}$$

★(1)

$$x_1(t) = x_m cos(\omega t + 54.9^{\circ})$$



$$x_2(t) = x_m cos(\omega t - 54.9^{\circ}) = x_m cos(\omega t + 360^{\circ} - 54.9^{\circ})$$

So only $\phi = -54.9^{\circ}$ will give the correct initial velocity.

★(2)

Or $v_x(t) = \frac{dx}{dt} = -\omega x_{\mathrm{m}} \sin(\omega t + \varphi)$

$$v_x(0) = -\omega x_m \sin\phi = -(1.035 m/s)\sin\phi$$

$= -0.847 m/s$ for $\phi_1 = 54.9^{\circ}$

$= 0.847 m/s$ for $\phi_2 = 305.1^{\circ}$

➡ $\phi_2 = 305.1^{\circ} = 5.33 radians$ is correct.

$$x(t) = (0.109m)\cos[9.54(rad/s)t + 5.33rad]$$

The simple pendulum

Fig(17-10) shows a simple pendulum of length *L* and particle mass *m*.

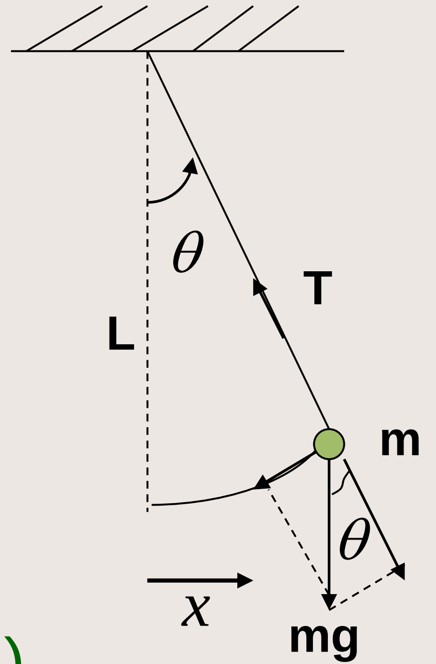
The restoring force is:

$$F_\tau = -mg\sin\theta \qquad (17\text{-}22)$$

If the θ is small, $\sin\theta \approx \theta$

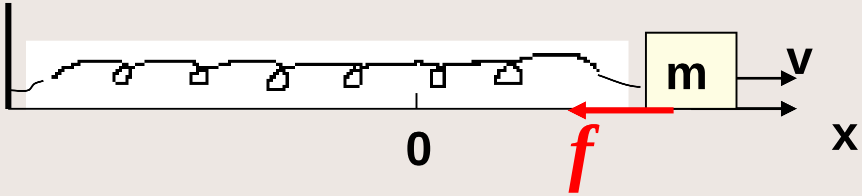
$$F_\tau \approx -mg\theta = -mg\frac{x}{L} = m\ddot{x} \qquad (17\text{-}23)$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}} = 2\pi\sqrt{\frac{L}{g}} \qquad (17\text{-}24)$$



Fig(17-10)

Damped harmonic motion



Up to this point we have assumed that no frictional force act on the system.

For real oscillator, there may be friction, air resistance act on the system, the amplitude will decrease.

1. This loss in amplitude is called "*damping*" and the motion is called "damped harmonic motion".

Fig17-16 compare the motion of undamped and damped oscillators.

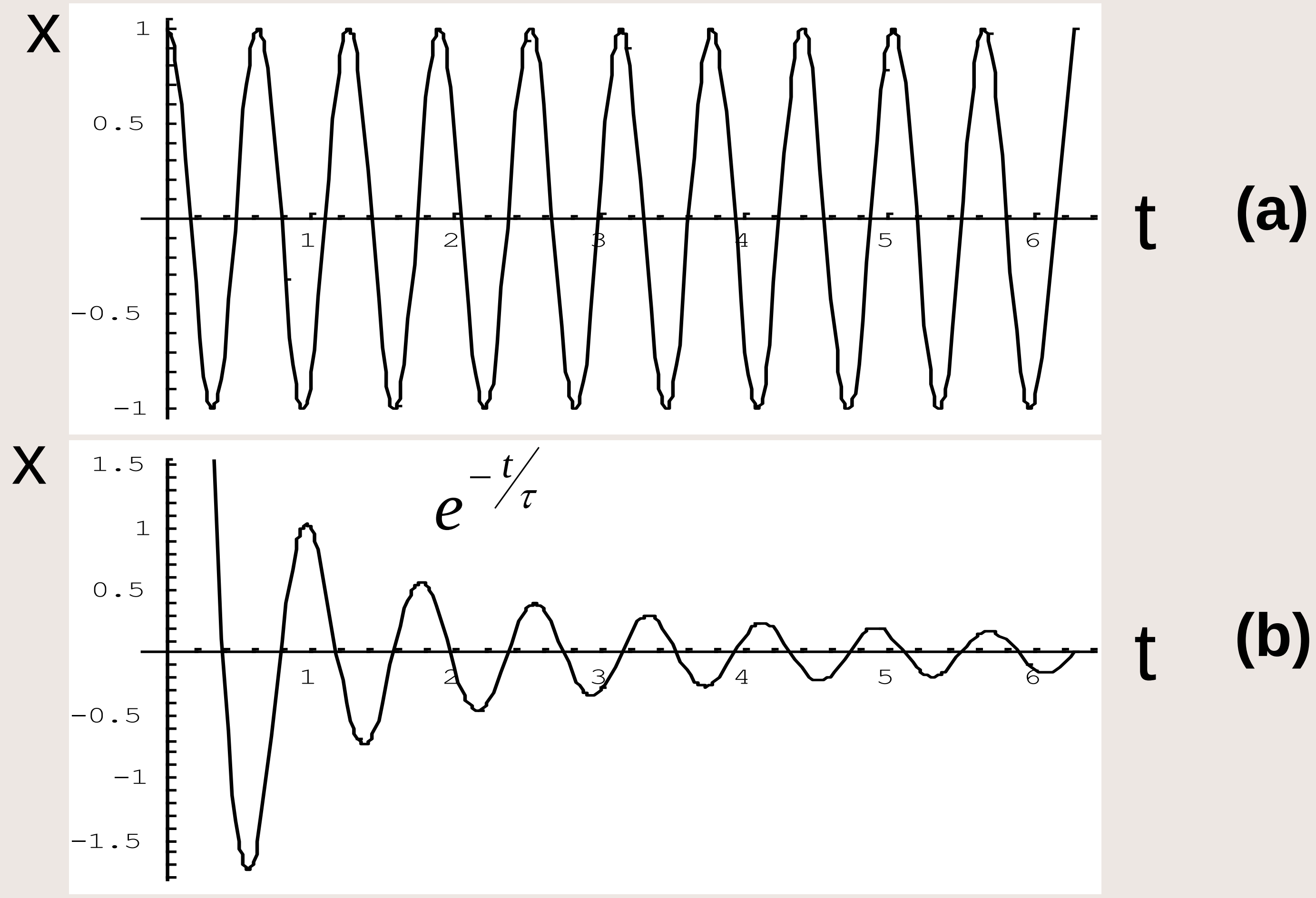


Fig 17-16

Forced oscillations and resonance

Forced oscillations:

Oscillations of a system carried out under the action of an external periodical force, such as

$$F_x(t) = F_m \sin \omega'' t$$

or a successive action of an external non-periodical force.

ω'' and ω ,

,Which frequency will the forced oscillation system take?

Forced oscillation system takes the frequency of the external force, namely ω'' .

Resonance:

The amplitude of the forced oscillation can increase much as ω'' approaches ω.

This condition is known as "resonance" and ω'' is called "resonant angular frequency".

In the case with damping, the rate at which energy is provided by the driving force exactly matches the rate at which energy is dissipated by the damping force.

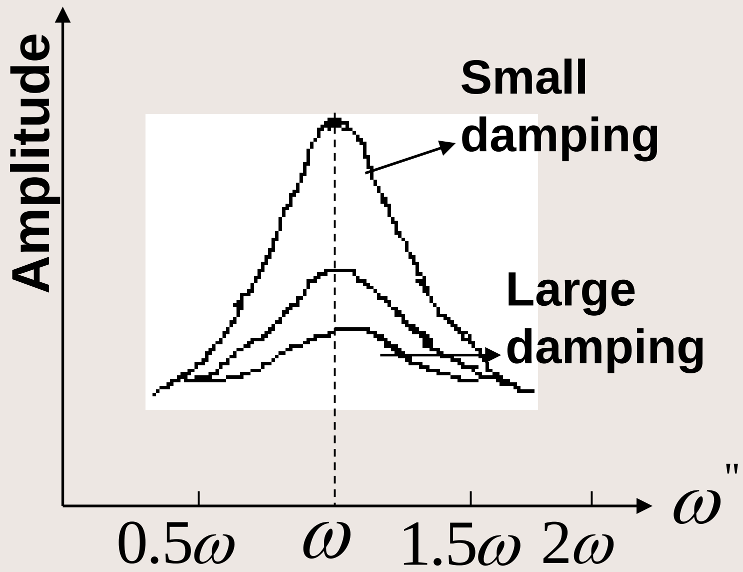


Fig 17-19